

L^1 -DINI CONDITIONS AND LIMITING BEHAVIOR OF WEAK TYPE ESTIMATES FOR SINGULAR INTEGRALS

YONG DING AND XUDONG LAI

ABSTRACT. In 2006, Janakiraman [10] showed that if Ω with mean value zero on \mathbb{S}^{n-1} satisfies the condition:

$$\sup_{|\xi|=1} \int_{\mathbb{S}^{n-1}} |\Omega(\theta) - \Omega(\theta + \delta\xi)| d\sigma(\theta) \leq Cn\delta \int_{\mathbb{S}^{n-1}} |\Omega(\theta)| d\sigma(\theta), \quad (*)$$

where $0 < \delta < \frac{1}{n}$, then for the singular integral operator T_Ω with homogeneous kernel, the following limiting behavior holds:

$$\lim_{\lambda \rightarrow 0_+} \lambda m(\{x \in \mathbb{R}^n : |T_\Omega f(x)| > \lambda\}) = \frac{1}{n} \|\Omega\|_1 \|f\|_1, \quad (**)$$

for $f \in L^1(\mathbb{R}^n)$ with $f \geq 0$.

In the present paper, we prove that if replacing the condition $(*)$ by more general condition, the L^1 -Dini condition, then the limiting behavior $(**)$ still holds for the singular integral T_Ω . In particular, we give an example which satisfies the L^1 -Dini condition, but does not satisfy $(*)$. Hence, we improve essentially the above result given in [10]. To prove our conclusion, we show that the L^1 -Dini conditions defined respectively via rotation and translation in \mathbb{R}^n are equivalent (see Theorem 2.5 below), which has its own interest in the theory of singular integrals. Moreover, similar limiting behavior for the fractional integral operator $T_{\Omega, \alpha}$ with homogeneous kernel is also established in this paper.

1. INTRODUCTION

Suppose that the function Ω defined on $\mathbb{R}^n \setminus \{0\}$ satisfies the following conditions:

$$(1.1) \quad \Omega(\lambda x) = \Omega(x), \quad \text{for any } \lambda > 0 \text{ and } x \in \mathbb{R}^n \setminus \{0\},$$

$$(1.2) \quad \int_{\mathbb{S}^{n-1}} \Omega(\theta) d\sigma(\theta) = 0$$

and $\Omega \in L^1(\mathbb{S}^{n-1})$, where \mathbb{S}^{n-1} denotes the unit sphere in \mathbb{R}^n and $d\sigma$ is the area measure on \mathbb{S}^{n-1} . Then the singular integral T_Ω with homogenous kernel is defined by

$$T_\Omega f(x) = \text{p.v.} \int \frac{\Omega(x-y)}{|x-y|^n} f(y) dy.$$

2010 *Mathematics Subject Classification.* 42B20.

Key words and phrases. Limiting behavior, weak type estimate, singular integral operator, L^1 -Dini condition.

The work is supported by NSFC (No.11371057, 11471033), SRFDP (No.20130003110003) and the Fundamental Research Funds for the Central Universities (No.2014KJJCA10).

Xudong Lai is the corresponding author.

It is well known that if Ω is odd and $\Omega \in L^1(\mathbb{S}^{n-1})$ (or Ω is even and $\Omega \in L \log^+ L(\mathbb{S}^{n-1})$), T_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ (see [2]), that is,

$$(1.3) \quad \|T_\Omega f\|_p \leq C_p \|f\|_p.$$

For $p = 1$, Seeger [12] showed that if $\Omega \in L \log^+ L(\mathbb{S}^{n-1})$,

$$(1.4) \quad m(\{x \in \mathbb{R}^n : |T_\Omega f(x)| > \lambda\}) \leq C_1 \frac{\|f\|_1}{\lambda}.$$

If Ω is an odd function, the usual Calderón-Zygmund method of rotation gives some information of the constant in (1.3). In fact, $C_p = \frac{\pi}{2} H_p \|\Omega\|_1$ (see [8]), where H_p denotes the L^p norm of the Hilbert transform ($1 < p < \infty$).

In 2004, Janakiraman [9] showed that the constants C_p in (1.3) and C_1 in (1.4) are at worst $C \log n \|\Omega\|_1$ if Ω satisfies (1.1), (1.2) and the following *regularity condition* :

$$(1.5) \quad \sup_{|\xi|=1} \int_{\mathbb{S}^{n-1}} |\Omega(\theta) - \Omega(\theta + \delta\xi)| d\sigma(\theta) \leq Cn\delta \int_{\mathbb{S}^{n-1}} |\Omega(\theta)| d\sigma(\theta), \quad 0 < \delta < \frac{1}{n},$$

where C is a constant independent of the dimension. In 2006, Janakiraman [10] extended further this result to the limiting case. Let μ be a signed measure on \mathbb{R}^n , which is absolutely continuous with respect to Lebesgue measure and $|\mu|(\mathbb{R}^n) < \infty$, here $|\mu|$ is the total variation of μ . Define

$$(1.6) \quad T_\Omega \mu(x) = \text{p.v.} \int \frac{\Omega(x-y)}{|x-y|^n} d\mu(y).$$

Theorem A ([10]). *Suppose Ω satisfies (1.1), (1.2) and the regularity condition (1.5). Then*

$$\lim_{\lambda \rightarrow 0_+} \lambda m(\{x \in \mathbb{R}^n : |T_\Omega \mu(x)| > \lambda\}) = \frac{1}{n} \|\Omega\|_1 |\mu(\mathbb{R}^n)|.$$

As a consequence of Theorem A, Janakiraman showed indeed that

Corollary A ([10]). *Let $f \in L^1(\mathbb{R}^n)$ and $f \geq 0$. Suppose Ω satisfies (1.1), (1.2) and (1.5), then*

$$(1.7) \quad \lim_{\lambda \rightarrow 0_+} \lambda m(\{x \in \mathbb{R}^n : |T_\Omega f(x)| > \lambda\}) = \frac{1}{n} \|\Omega\|_1 \|f\|_1.$$

The limiting behavior (1.7) is very interesting since it gives some information of the best constant for weak type (1,1) estimate of the homogeneous singular integral operator T_Ω in some sense. However, note that the condition (1.5) seems to be strong compared with the *Hörmander condition* (see also [13]):

$$(1.8) \quad \sup_{y \neq 0} \int_{|x| > 2|y|} |K(x-y) - K(x)| dx < \infty,$$

where K is the kernel of the Calderón-Zygmund singular integral operator. Hence, it is natural to ask whether (1.7) still holds if replacing (1.5) by the Hörmander condition (1.8)? The purpose of this paper is to give an affirmative answer to the above problem for the case of $K(x) = \Omega(x)|x|^{-n}$.

Before stating our results, we give the definition of the L^1 -Dini condition.

Definition 1.1 (L^1 -Dini condition). Let Ω satisfy (1.1). We say that Ω satisfies the L^1 -Dini condition if:

- (i) $\Omega \in L^1(\mathbb{S}^{n-1})$;
- (ii) $\int_0^1 \frac{\omega_1(\delta)}{\delta} d\delta < \infty$, where ω_1 denotes the L^1 integral modulus of continuity of Ω defined by

$$\omega_1(\delta) = \sup_{\|\rho\| \leq \delta} \int_{\mathbb{S}^{n-1}} |\Omega(\rho\theta) - \Omega(\theta)| d\sigma(\theta),$$

where ρ is a rotation on \mathbb{R}^n and $\|\rho\| := \sup\{|\rho x' - x'| : x' \in \mathbb{S}^{n-1}\}$.

Let us recall two important facts in [1] and [3].

Lemma A ([1]). *If Ω satisfies the L^1 -Dini condition, then $\Omega \in L \log^+ L(\mathbb{S}^{n-1})$ and $K(x) = \Omega(x)|x|^{-n}$ satisfies the Hörmander condition (1.8).*

Lemma B ([3]). *If $K(x) = \Omega(x)|x|^{-n}$ satisfies the Hörmander condition (1.8), then $\Omega \in L \log^+ L(\mathbb{S}^{n-1})$ and Ω satisfies the L^1 -Dini condition.*

By Lemma A and Lemma B, one can see immediately that for the kernel $K(x) = \Omega(x)|x|^{-n}$ the Hörmander condition (1.8) is equivalent to the L^1 -Dini condition.

In Section 2, we will prove that the regularity condition (1.5) is stronger than the L^1 -Dini condition (see Proposition 2.1). Also we will give an example to show that the L^1 -Dini condition is rigorously weaker than the regularity condition (1.5) (see Example 2.2).

Our main result in this paper is to prove that the limiting behavior (1.7) still holds if replacing the condition (1.5) by the L^1 -Dini condition.

Theorem 1.2. *Suppose Ω satisfies (1.1), (1.2) and the L^1 -Dini condition. Let μ be an absolutely continuous signed measure on \mathbb{R}^n with respect to Lebesgue measure and $|\mu|(\mathbb{R}^n) < \infty$. Let T_Ω be defined by (1.6). Then we have*

$$(1.9) \quad \lim_{\lambda \rightarrow 0_+} \lambda m(\{x \in \mathbb{R}^n : |T_\Omega \mu(x)| > \lambda\}) = \frac{1}{n} \|\Omega\|_1 |\mu(\mathbb{R}^n)|.$$

By setting $\mu(E) = \int_E f(x) dx$ with $f \in L^1(\mathbb{R}^n)$ in Theorem 1.2, we have the following result.

Corollary 1.3. *Let $f \in L^1(\mathbb{R}^n)$ and $f \geq 0$. Suppose Ω satisfies (1.1), (1.2) and the L^1 -Dini condition. Then we have*

$$\lim_{\lambda \rightarrow 0_+} \lambda m(\{x \in \mathbb{R}^n : |T_\Omega f(x)| > \lambda\}) = \frac{1}{n} \|\Omega\|_1 \|f\|_1.$$

The next results are related to the limiting behavior for weak type estimate of the homogeneous fractional integral operator $T_{\Omega, \alpha}$, which is defined as

$$T_{\Omega, \alpha} f(x) = \int \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y) dy, \quad 0 < \alpha < n.$$

It is well known that the fractional integral operator $T_{\Omega, \alpha}$, a generalization of Riesz potential, has been studied by many people (see the book [11] and the references therein). In [5], while studying

the boundedness of $T_{\Omega, \alpha}$ on Hardy space, Ding and Lu introduced the following regularity condition of Ω :

$$(1.10) \quad \int_0^1 \frac{\omega_q(\delta)}{\delta^{1+\alpha}} d\delta < \infty,$$

where ω_q denotes the L^q integral modulus of continuity of Ω .

To study the limiting behavior of the fractional operator with homogeneous kernel, we need some regularity conditions similar to (1.10). For convenience, we give the following notation.

Definition 1.4 (L_α^s -Dini condition). Let Ω satisfy (1.1), $1 \leq s \leq \infty$ and $0 < \alpha < n$. We say that Ω satisfies the L_α^s -Dini condition if

- (i) $\Omega \in L^s(\mathbb{S}^{n-1})$;
- (ii) $\int_0^1 \frac{\omega_1(\delta)}{\delta^{1+\alpha}} d\delta < \infty$, where ω_1 is defined as that in Definition 1.1.

Let ν be an absolutely continuous signed measure on \mathbb{R}^n with respect to Lebesgue measure and $|\nu|(\mathbb{R}^n) < \infty$. Define

$$(1.11) \quad T_{\Omega, \alpha} \nu(x) = \int \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} d\nu(y).$$

We have the following results for $T_{\Omega, \alpha}$.

Theorem 1.5. *Let ν be an absolutely continuous signed measure on \mathbb{R}^n with respect to Lebesgue measure and $|\nu|(\mathbb{R}^n) < \infty$. Let $0 < \alpha < n$ and $r = \frac{n}{n-\alpha}$. Suppose Ω satisfies (1.1), (1.2) and the L_α^r -Dini condition. Then*

$$\lim_{\lambda \rightarrow 0_+} \lambda^r m(\{x \in \mathbb{R}^n : |T_{\Omega, \alpha} \nu(x)| > \lambda\}) = \frac{1}{n} \|\Omega\|_r^r |\nu(\mathbb{R}^n)|^r.$$

Corollary 1.6. *Let $0 < \alpha < n$ and $r = \frac{n}{n-\alpha}$. Let $f \in L^1(\mathbb{R}^n)$ and $f \geq 0$. Suppose Ω satisfies (1.1), (1.2) and the L_α^r -Dini condition. Then we have*

$$\lim_{\lambda \rightarrow 0_+} \lambda^r m(\{x \in \mathbb{R}^n : |T_{\Omega, \alpha} f(x)| > \lambda\}) = \frac{1}{n} \|\Omega\|_r^r \|f\|_1^r.$$

We would like to point out the proof of Theorem 1.2 follows the idea from [10]. However, to establish the limiting behavior of the singular integral operator T_Ω with Ω satisfying the L^1 -Dini condition, we need study carefully the regularity of Ω . More precisely, we will show that two different L^1 -Dini conditions are equivalent (see Theorem 2.5).

The paper is organized as follows. In Section 2, we give some properties of the L^1 -Dini condition and the embedding relation between the regularity condition (1.5) and the L^1 -Dini condition. An example which shows the L^1 -Dini condition is weaker than the condition (1.5) is also given in this section. The proof of Theorem 1.2 is given in Section 3. The outline of the proof of Theorem 1.5 is given in final section. Throughout this paper the letter C will stand for a positive constant not necessarily the same one in each occurrence.

2. L^1 -DINI CONDITION

In this section, we discuss some properties of the L^1 -Dini condition. We first show that the regularity condition (1.5) is stronger than the L^1 -Dini condition.

Proposition 2.1. *If Ω satisfies (1.1), (1.2) and the condition (1.5), then Ω satisfies L^1 -Dini condition.*

Proof. We first claim that if Ω satisfies (1.1), (1.2) and (1.5), then there exists $C > 0$ such that

$$(2.1) \quad \omega_1(\delta) \leq \sup_{|\xi|=1} \int_{\mathbb{S}^{n-1}} |\Omega(\theta + C\delta\xi) - \Omega(\theta)| d\theta$$

for any $0 < \delta < \frac{1}{2}$. To prove (2.1), by Definition 1.1, it is enough to show that for any fixed $\theta \in \mathbb{S}^{n-1}$,

$$\{\rho\theta : \|\rho\| \leq \delta\} \subset \left\{ \frac{\theta + C\delta\xi}{|\theta + C\delta\xi|} : \xi \in \mathbb{S}^{n-1} \right\}$$

for some constant $C > 0$. For convenience, let

$$A = \{\rho\theta : \|\rho\| \leq \delta\}$$

and

$$B(C) = \left\{ \frac{\theta + C\delta\xi}{|\theta + C\delta\xi|} : \xi \in \mathbb{S}^{n-1} \right\}.$$

Thus $A = \{\eta \in \mathbb{S}^{n-1} : |\eta - \theta| \leq \delta\}$. Choose $C = 2$, we will show that

$$(2.2) \quad B(2) \supset A.$$

Notice that the function $f(\xi) = \left| \frac{\theta + 2\delta\xi}{|\theta + 2\delta\xi|} - \theta \right|$ is continuous on \mathbb{S}^{n-1} . Since \mathbb{S}^{n-1} is compact, then $f(\xi)$ can get its maximal value at a point of \mathbb{S}^{n-1} . Suppose ξ_0 is such a point that $f(\xi)$ get its maximal value at ξ_0 . Since $f(\theta) = 0$ and $f(-\theta) = 0$, ξ_0 must be located between θ and $-\theta$. Therefore again by the continuity of $f(\xi)$,

$$B(2) = \{\eta \in \mathbb{S}^{n-1} : |\eta - \theta| \leq \gamma\} \quad \text{with} \quad \gamma = f(\xi_0).$$

So to prove (2.2), it suffices to show that $\gamma \geq \delta$. By rotation, we may suppose $\theta = (1, 0, 0, \dots, 0)$. Choose $\xi = (0, 1, 0, \dots, 0)$. Then

$$\gamma \geq \left| \frac{\theta + 2\delta\xi}{|\theta + 2\delta\xi|} - \theta \right| = \left(2 - \frac{2}{\sqrt{1 + 4\delta^2}} \right)^{\frac{1}{2}} \geq \delta.$$

Hence we prove (2.1) by choosing $C = 2$.

Now we split the integral $\int_0^1 \frac{\omega_1(\delta)}{\delta} d\delta$ into two parts:

$$\int_0^{\frac{1}{2^n}} \frac{\omega_1(\delta)}{\delta} d\delta + \int_{\frac{1}{2^n}}^1 \frac{\omega_1(\delta)}{\delta} d\delta.$$

For the first integral, using estimate (2.1) and the regularity condition (1.5), we can get the bound $C\|\Omega\|_1$. For the second integral, using $\omega_1(\delta) \leq 2\|\Omega\|_1$ for any $0 < \delta < 1$, we can also get the bound $C\|\Omega\|_1$. Combining these, the proof is completed. \square

In the following, we give an example which satisfies (1.1), (1.2) and the L^1 -Dini condition but does not satisfy the regularity condition (1.5).

Example 2.2. Consider dimension $n = 2$, in this case, we denote $\mathbb{S}^1 = \{\theta : 0 \leq \theta \leq 2\pi\}$, where θ is the arc length on the unit circle. Let $\Omega(\theta) = \theta^{-\frac{1}{2}} - (\frac{2}{\pi})^{\frac{1}{2}}$. It can be easily extended to the whole space \mathbb{R}^2 such that Ω is homogeneous of degree zero.

By using representation of the differential of arc length, the integral of Ω on \mathbb{S}^1 can be rewritten as

$$\int_0^{2\pi} \Omega(\theta) d\theta,$$

where θ is again the arc length. Obviously, Ω in Example 2.2 satisfies (1.2).

Now let us first show that Ω in Example 2.2 does not satisfy the regularity condition (1.5). In fact, let δ be small enough. In two dimension, for any rotation $\|\rho\| \leq \delta$, we have $\rho\theta = \theta \pm s$, where $s = \|\rho\|$. For the case $\rho\theta = \theta + s$, we have

$$\begin{aligned} \int_0^{2\pi} |\Omega(\rho\theta) - \Omega(\theta)| d\theta &= \int_0^{2\pi-s} \left| \frac{1}{\theta^{1/2}} - \frac{1}{(\theta+s)^{1/2}} \right| d\theta \\ &\quad + \int_{2\pi-s}^{2\pi} \left| \frac{1}{\theta^{1/2}} - \frac{1}{(\theta+s-2\pi)^{1/2}} \right| d\theta \\ &= 4((2\pi-s)^{1/2} - (2\pi)^{1/2} + s^{1/2}) =: g(s), \end{aligned}$$

where in the first equality we use the fact that when $\theta \in (2\pi-s, 2\pi)$, $\rho\theta$ falls into $(0, s)$. A similar computation shows that if $\rho\theta = \theta - s$,

$$\int_0^{2\pi} |\Omega(\rho\theta) - \Omega(\theta)| d\theta = g(s).$$

It is not difficult to see that $g(s)$ is an increased function for $s \in [0, \delta]$ and $g(0) = 0$. Therefore we have

$$\omega_1(\delta) = \sup_{\|\rho\| \leq \delta} \int_0^{2\pi} |\Omega(\rho\theta) - \Omega(\theta)| d\theta = g(\delta).$$

Now by (2.1) in Lemma 2.1 (note that constant $C = 2$), we have

$$\begin{aligned} \frac{1}{2\delta} \sup_{|\xi|=1} \int_{\mathbb{S}^1} |\Omega(\theta + 2\delta\xi) - \Omega(\theta)| d\theta &\geq \frac{1}{2\delta} \omega_1(\delta) \\ &= 2 \left(\frac{1}{\delta^{1/2}} - \frac{(2\pi)^{1/2} - (2\pi - \delta)^{1/2}}{\delta} \right) \rightarrow +\infty \end{aligned}$$

as $\delta \rightarrow 0$. This means that Ω does not satisfy the regularity condition (1.5). By a direct computation, we have

$$\int_0^1 \frac{\omega_1(\delta)}{\delta} d\delta = 4 \int_0^1 \left(\frac{1}{\delta^{1/2}} - \frac{(2\pi)^{1/2} - (2\pi - \delta)^{1/2}}{\delta} \right) d\delta < \infty$$

and

$$\int_0^{2\pi} |\Omega(\theta)| d\theta < \infty$$

which means that Ω satisfies the L^1 -Dini condition in Definition 1.1.

In order to prove Theorem 1.2, we need to give an equivalent definition of the L^1 -Dini condition in Definition 1.1.

Recall in Definition 1.1, the L^1 -Dini condition is defined by the L^1 integral modulus ω_1 , and ω_1 is defined by ROTATION in \mathbb{R}^n . In [1], Calderón, Weiss and Zygmund gave another L^1 integral modulus $\tilde{\omega}_1$ which is defined by TRANSLATION in \mathbb{R}^n . Let Ω satisfy (1.1) and $\Omega \in L^1(\mathbb{S}^{n-1})$. Define $\tilde{\omega}_1$ as

$$(2.3) \quad \tilde{\omega}_1(\delta) = \sup_{|h| \leq \delta} \int_{\mathbb{S}^{n-1}} |\Omega(x' + h) - \Omega(x')| d\sigma(x'),$$

where $h \in \mathbb{R}^n$. Similarly, one may define the L^1 -Dini condition by the L^1 integral modulus $\tilde{\omega}_1$.

Definition 2.3. Let Ω satisfy (1.1). It is said that Ω satisfies the L^1 -Dini condition if:

- (i) $\Omega \in L^1(\mathbb{S}^{n-1})$;
- (ii) $\int_0^1 \frac{\tilde{\omega}_1(\delta)}{\delta} d\delta < \infty$, where $\tilde{\omega}_1(\delta)$ is defined by (2.3).

As it is pointed out in [1], the L^1 -Dini condition in Definition 1.1 is the most natural one. However, in some cases, the L^1 -Dini definition in Definition 2.3 is more convenient in application. Thus, a natural problem is that, is there any relationship between those two kind of L^1 -Dini conditions defined by Definition 1.1 and Definition 2.3, respectively.

Below we will show that these two kind L^1 -Dini conditions defined respectively by Definition 1.1 and Definition 2.3 are equivalent indeed. Let us first recall a useful lemma.

Lemma 2.4 (see Lemma 5 in [1]). *There exist positive constants α_0, C depending only on the dimension n such that if Ω is any function integrable over \mathbb{S}^{n-1} and $0 < |h| \leq \alpha_0$, $h \in \mathbb{R}^n$, then*

$$(2.4) \quad \int_{\mathbb{S}^{n-1}} |\Omega(\xi - h) - \Omega(\xi)| d\sigma(\xi) \leq C \sup_{\|\rho\| \leq |h|} \int_{\mathbb{S}^{n-1}} |\Omega(\rho\xi) - \Omega(\xi)| d\sigma(\xi).$$

Note that we may choose the constant α_0 in Lemma 2.4 less than 1.

Theorem 2.5. *L^1 -Dini conditions defined respectively in Definition 1.1 and Definition 2.3 are equivalent.*

Proof. By Definition 1.1 and Definition 2.3, it is enough to show that for $\Omega \in L^1(\mathbb{S}^{n-1})$, the following condition (a) and (b) are equivalent:

- (a) $\int_0^1 \frac{\omega_1(\delta)}{\delta} d\delta < \infty$, where $\omega_1(\delta) = \sup_{\|\rho\| \leq \delta} \int_{\mathbb{S}^{n-1}} |\Omega(\rho x') - \Omega(x')| d\sigma(x')$,
- (b) $\int_0^1 \frac{\tilde{\omega}_1(\delta)}{\delta} d\delta < \infty$, where $\tilde{\omega}_1(\delta) = \sup_{|h| \leq \delta} \int_{\mathbb{S}^{n-1}} |\Omega(x' + h) - \Omega(x')| d\sigma(x')$.

We first show that (b) implies (a). By (2.1) (note that the constant $C = 2$), we have

$$\omega_1(\delta) \leq \sup_{|\xi|=1} \int_{\mathbb{S}^{n-1}} |\Omega(\theta + 2\delta\xi) - \Omega(\theta)| d\sigma(\theta) \leq \tilde{\omega}_1(2\delta).$$

Hence we have

$$\begin{aligned} \int_0^1 \frac{\omega_1(\delta)}{\delta} d\delta &= \left(\int_0^{1/2} + \int_{1/2}^1 \right) \frac{\omega_1(\delta)}{\delta} d\delta \leq \int_0^{1/2} \frac{\tilde{\omega}_1(2\delta)}{\delta} d\delta + \int_{1/2}^1 \frac{\omega_1(\delta)}{\delta} d\delta \\ &\leq \int_0^1 \frac{\tilde{\omega}_1(\delta)}{\delta} d\delta + C\|\Omega\|_1. \end{aligned}$$

Now we turn to the other part: (a) implies (b). By Lemma 2.4, there exists a constant $0 < a_0 < 1$ such that for any $0 < |h| \leq a_0$, we have

$$\int_{\mathbb{S}^{n-1}} |\Omega(\xi + h) - \Omega(\xi)| d\sigma(\xi) \leq C \sup_{\|\rho\| \leq |h|} \int_{\mathbb{S}^{n-1}} |\Omega(\rho\theta) - \Omega(\rho)| d\sigma(\theta).$$

If $0 < \delta < a_0$, then

$$\begin{aligned} \tilde{\omega}_1(\delta) &= \sup_{|h| \leq \delta} \int_{\mathbb{S}^{n-1}} |\Omega(\xi + h) - \Omega(\xi)| d\sigma(\xi) \\ &\leq C \sup_{|h| \leq \delta} \sup_{\|\rho\| \leq |h|} \int_{\mathbb{S}^{n-1}} |\Omega(\rho\theta) - \Omega(\theta)| d\sigma(\theta) \leq C\omega_1(\delta). \end{aligned}$$

If $a_0 \leq \delta < 1$, we get

$$\tilde{\omega}_1(\delta) = \sup_{|h| \leq \delta} \int_{\mathbb{S}^{n-1}} |\Omega(\theta + h) - \Omega(\theta)| d\sigma(\theta) \leq \|\Omega\|_1 + \sup_{|h| \leq \delta} \int_{\mathbb{S}^{n-1}} |\Omega(\theta + h)| d\sigma(\theta).$$

If we can prove that

$$(2.5) \quad \sup_{|h| \leq \delta} \int_{\mathbb{S}^{n-1}} |\Omega(\theta + h)| d\sigma(\theta) \leq C\|\Omega\|_1,$$

then we have

$$\begin{aligned} \int_0^1 \frac{\tilde{\omega}_1(\delta)}{\delta} d\delta &= \left(\int_0^{a_0} + \int_{a_0}^1 \right) \frac{\tilde{\omega}_1(\delta)}{\delta} d\delta \leq C \int_0^1 \frac{\omega_1(\delta)}{\delta} d\delta + \int_{a_0}^1 \frac{\tilde{\omega}_1(\delta)}{\delta} d\delta \\ &\leq C \int_0^1 \frac{\omega_1(\delta)}{\delta} d\delta + \int_{a_0}^1 \frac{1}{\delta} \left(\|\Omega\|_1 + \sup_{|h| \leq \delta} \int_{\mathbb{S}^{n-1}} |\Omega(\theta + h)| d\sigma(\theta) \right) d\delta \\ &\leq C \int_0^1 \frac{\omega_1(\delta)}{\delta} d\delta + C\|\Omega\|_1. \end{aligned}$$

Hence, to complete the proof of Theorem 2.5, it remains to verify (2.5). By rotation, we may assume that $h = (h_1, 0, \dots, 0)$, where $0 < h_1 < 1$. By using the spherical coordinate formula on \mathbb{S}^{n-1} (see Appendix D in [7]), we can write

$$\begin{aligned} (2.6) \quad \int_{\mathbb{S}^{n-1}} \left| \Omega\left(\frac{x+h}{|x+h|}\right) \right| d\sigma(x) &= \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} \left| \Omega\left(\frac{x(\varphi)+h}{|x(\varphi)+h|}\right) \right| \\ &\quad \times |J(n, \varphi)| d\varphi_{n-1} \cdots d\varphi_1, \end{aligned}$$

where $x(\varphi)$ and $J(n, \varphi)$ are defined as

$$\begin{aligned} x_1 &= \cos \varphi_1, \\ x_2 &= \sin \varphi_1 \cos \varphi_2, \\ x_3 &= \sin \varphi_1 \sin \varphi_2 \cos \varphi_3, \\ &\dots \\ x_{n-1} &= \sin \varphi_1 \sin \varphi_2 \cdots \sin \varphi_{n-2} \cos \varphi_{n-1}, \\ x_n &= \sin \varphi_1 \sin \varphi_2 \cdots \sin \varphi_{n-2} \sin \varphi_{n-1}; \\ J(n, \varphi) &= (\sin \varphi_1)^{n-2} \cdots (\sin \varphi_{n-3})^2 \sin \varphi_{n-2}. \end{aligned}$$

Compared with $x(\varphi)$, $\frac{x(\varphi)+h}{|x(\varphi)+h|}$ can be written as $x(\theta)$ with $\theta_i = \varphi_i$, $2 \leq i \leq n-1$. This can be seen from the point of geometry since $h = (h_1, 0, \dots, 0)$. Hence we make a variable transform that maps $(\varphi_1, \varphi_2, \dots, \varphi_{n-1})$ into $(\theta_1, \theta_2, \dots, \theta_{n-1})$ such that

$$\left\{ \begin{array}{ll} \frac{\cos \varphi_1 + h_1}{\sqrt{1+2h_1 \cos \varphi_1 + h_1^2}} &= \cos \theta_1, \quad \frac{\sin \varphi_1}{\sqrt{1+2h_1 \cos \varphi_1 + h_1^2}} = \sin \theta_1, \\ \varphi_2 &= \theta_2, \\ &\dots \\ \varphi_{n-1} &= \theta_{n-1}. \end{array} \right.$$

Thus $\frac{x(\varphi)+h}{|x(\varphi)+h|} = x(\theta)$. It is easy to see

$$\tan \theta_1 = \frac{\sin \varphi_1}{\cos \varphi_1 + h_1}.$$

Then we have

$$d\theta_1 = \left(\arctan \frac{\sin \varphi_1}{\cos \varphi_1 + h_1} \right)' d\varphi_1 = \frac{1 + h_1 \cos \varphi_1}{1 + 2h_1 \cos \varphi_1 + h_1^2} d\varphi_1.$$

Note that $0 \leq \varphi_1 \leq \pi$ and $0 < h_1 < 1$, then $0 < \theta_1 < \pi$. Therefore the right side of (2.6) is bounded by

$$\begin{aligned} &\int_{\theta_1=0}^{\pi} \cdots \int_{\theta_{n-2}=0}^{\pi} \int_{\theta_{n-1}=0}^{2\pi} |\Omega(x(\theta))| |J(n, \theta)| \frac{(1 + 2 \cos \varphi_1 h_1 + h_1^2)^{n/2}}{1 + h_1 \cos \varphi_1} d\theta_{n-1} \cdots d\theta_1 \\ &\leq 2^{n-1} \int_{\theta_1=0}^{\pi} \cdots \int_{\theta_{n-2}=0}^{\pi} \int_{\theta_{n-1}=0}^{2\pi} |\Omega(x(\theta))| |J(n, \theta)| d\theta_{n-1} \cdots d\theta_1 \\ &= 2^{n-1} \int_{\mathbb{S}^{n-1}} |\Omega(x)| d\sigma(x), \end{aligned}$$

where in the first inequality we use

$$\frac{1 + 2h_1 \cos \varphi_1 + h_1^2}{1 + h_1 \cos \varphi_1} \leq 2$$

and $0 < h_1 < 1$. Therefore we finish the proof of (2.5). \square

Remark 2.6. By Theorem 2.5, when applying the L^1 -Dini condition, one may use its definition in Definition 1.1 or Definition 2.3 according to the request of application.

The L_α^r -Dini condition that we introduce in Definition 1.4 is defined by rotation. It is natural to consider the translation version.

Definition 2.7. Let Ω satisfy (1.1), $1 \leq s \leq \infty$ and $0 < \alpha < n$. We say that Ω satisfies the L_α^s -Dini condition if

- (i) $\Omega \in L^s(\mathbb{S}^{n-1})$;
- (ii) $\int_0^1 \frac{\tilde{\omega}_1(\delta)}{\delta^{1+\alpha}} d\delta < \infty$, where $\tilde{\omega}_1$ is defined by (2.3).

By using the similar way that we prove Theorem 2.5, we have the following result.

Theorem 2.8. Let $s \geq 1$ and $0 < \alpha < n$. L_α^s -Dini conditions defined respectively in Definition 1.4 and Definition 2.7 are equivalent.

3. PROOF OF THEOREM 1.2

In this section we give the proof of Theorem 1.2. Suppose μ is a signed measure on \mathbb{R}^n . For $t > 0$, let $\mu_t(E) = \mu(\frac{E}{t})$, where E is the Lebesgue measurable set in \mathbb{R}^n .

3.1. Some elementary facts. Let us begin with some elementary facts.

Lemma 3.1. Let μ be a signed measure on \mathbb{R}^n . Suppose E is the μ_t measurable set. Then

$$|\mu_t|(E) = |\mu|_t(E).$$

Proof. Since μ is a signed measure on \mathbb{R}^n , by the Hahn decomposition (see [6]), there exists a positive set P and a negative set N such that $P \cup N = \mathbb{R}^n$ and $P \cap N = \emptyset$. If P' and N' are another such pair, then $P \triangle P' (= N \triangle N')$ is null for μ . Therefore $\mu^+(E) = \mu(E \cap P)$ and $\mu^-(E) = -\mu(E \cap N)$. Since the Hahn decomposition is unique, the pair tP and tN can be seen as the Hahn decomposition of μ_t . Then for any μ_t measurable set E , we have

$$\begin{aligned} |\mu_t|(E) &= (\mu_t)^+(E) + (\mu_t)^-(E) = \mu_t(E \cap tP) - \mu_t(E \cap tN) \\ &= \mu\left(\frac{1}{t}E \cap P\right) - \mu\left(\frac{1}{t}E \cap N\right) \\ &= |\mu|\left(\frac{1}{t}E\right) = |\mu|_t(E). \end{aligned}$$

Hence the proof is completed. □

Lemma 3.2. Let μ be a nonnegative measure defined on \mathbb{R}^n and $\mu(\mathbb{R}^n) = 1$. Suppose μ is absolutely continuous with respect to Lebesgue measure. Then for any $0 < \varepsilon < 1$, there exists a_ε , $0 < a_\varepsilon < \infty$, such that $\mu(B(0, a_\varepsilon)) = \varepsilon$.

Proof. Since $\mu(\mathbb{R}^n) = 1$, there exists M , $0 < M < \infty$, such that $\mu(B(0, M)) \geq \varepsilon$.

Set $A_\varepsilon = \{r : \mu(B(0, r)) \geq \varepsilon\}$ and denote $a_\varepsilon = \inf_{r \in A_\varepsilon} r$. It is easy to see that $a_\varepsilon \leq M < \infty$. We claim that $\mu(B(0, a_\varepsilon)) = \varepsilon$. In fact, by the definition of infimum, for any $\alpha > 0$, there exists a $r \in A_\varepsilon$, which satisfies $a_\varepsilon < r < a_\varepsilon + \alpha$, such that $\mu(B(0, r)) \geq \varepsilon$. Hence

$$\mu(B(0, a_\varepsilon)) \geq \mu(B(0, r)) - \mu(B(0, r) \setminus B(0, a_\varepsilon)) \geq \varepsilon - \mu(B(0, a_\varepsilon + \alpha) \setminus B(0, a_\varepsilon)).$$

Note that $m(B(0, a_\varepsilon + \alpha) \setminus B(0, a_\varepsilon)) \rightarrow 0$ as $\alpha \rightarrow 0$. Since μ is absolutely continuous with respect to Lebesgue measure, so $\mu(B(0, a_\varepsilon + \alpha) \setminus B(0, a_\varepsilon)) \rightarrow 0$ as $\alpha \rightarrow 0$. Hence $\mu(B(0, a_\varepsilon)) \geq \varepsilon$.

On the other hand, by the definition of a_ε , for any $0 < r < a_\varepsilon$, we have $\mu(B(0, r)) < \varepsilon$. Note

$$\mu(B(0, a_\varepsilon)) \leq \mu(B(0, r)) + \mu(B(0, a_\varepsilon) \setminus B(0, r)) < \varepsilon + \mu(B(0, a_\varepsilon) \setminus B(0, r)).$$

Since $\mu(B(0, a_\varepsilon) \setminus B(0, r)) \rightarrow 0$ as $r \rightarrow a_\varepsilon$, then $\mu(B(0, a_\varepsilon)) \leq \varepsilon$. Therefore we finish the proof. \square

Lemma 3.3. *Let $0 \leq \alpha < n$ and $r = \frac{n}{n-\alpha}$. For a fixed $\lambda > 0$, we have*

$$(3.1) \quad \lambda^r m\left(\left\{x \in \mathbb{R}^n : \frac{|\Omega(x)|}{|x|^{n-\alpha}} > \lambda\right\}\right) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} |\Omega(\theta)|^r d\sigma(\theta).$$

Proof. By making a polar transform,

$$\begin{aligned} m\left(\left\{x \in \mathbb{R}^n : \frac{|\Omega(x)|}{|x|^{n-\alpha}} > \lambda\right\}\right) &= \int_{\mathbb{S}^{n-1}} \int_0^\infty \chi_{\{|\Omega(\theta)|/s^{n-\alpha} > \lambda\}} s^{n-1} ds d\sigma(\theta) \\ &= \int_{\mathbb{S}^{n-1}} \int_0^{(|\Omega(\theta)|/\lambda)^{\frac{1}{n-\alpha}}} s^{n-1} ds d\sigma(\theta) \\ &= \frac{1}{n \cdot \lambda^r} \int_{\mathbb{S}^{n-1}} |\Omega(\theta)|^r d\sigma(\theta). \end{aligned}$$

\square

Lemma 3.4. *Let μ be a absolutely continuous signed measure on \mathbb{R}^n with respect to Lebesgue measure and $|\mu|(\mathbb{R}^n) < \infty$. Suppose Ω satisfies (1.1), (1.2) and the L^1 -Dini condition. For any $\lambda > 0$, we have*

$$(3.2) \quad \lambda m(\{x \in \mathbb{R}^n : |T_\Omega \mu(x)| > \lambda\}) \leq C |\mu|(\mathbb{R}^n)$$

where the constant C only depends on Ω and the dimension.

Proof. Since μ is a absolutely continuous signed measure on \mathbb{R}^n with respect to Lebesgue measure and $|\mu|(\mathbb{R}^n) < \infty$, by the Radon-Nikodym's theorem (see [6]), there exists a integrable function f such that $d\mu(x) = f(x)dx$. Therefore we have

$$T_\Omega \mu(x) = T_\Omega f(x).$$

Now the rest of the proof can be found in the book [7]. By carefully examining the proof there, the weak (1,1) bound in (3.2) is $C(\|\Omega\|_1 + \int_0^1 \frac{\omega_1(s)}{s} ds)$. \square

3.2. Key lemma. Now we give a lemma which plays a key role in the proof of Theorem 1.2.

Lemma 3.5. *Let μ be an absolutely continuous signed measure with respect to Lebesgue measure on \mathbb{R}^n and $|\mu|(\mathbb{R}^n) < +\infty$. Suppose Ω satisfies (1.1), (1.2) and the L^1 -Dini condition. Let T_Ω be defined by (1.6). Then we have*

$$(3.3) \quad \lim_{t \rightarrow 0_+} \lambda m(\{x \in \mathbb{R}^n : |T_\Omega \mu_t(x)| > \lambda\}) = \frac{1}{n} \|\Omega\|_1 |\mu|(\mathbb{R}^n)$$

for any $\lambda > 0$.

Proof. Without loss of generality, we may assume $|\mu|(\mathbb{R}^n) = 1$. Let δ is small enough such that $0 < \delta \ll 1$. For any fixed $\lambda > 0$, choose ε such that $0 < \varepsilon \leq \frac{1}{2}\delta\lambda$. By Lemma 3.2, there exists an a_ε with $0 < a_\varepsilon < \infty$, such that $|\mu|(B(0, a_\varepsilon)) = 1 - \varepsilon$. Set $\varepsilon_t = a_\varepsilon \cdot t$, by Lemma 3.1 we have

$$|\mu_t|(B(0, \varepsilon_t)) = |\mu|_t(B(0, \varepsilon_t)) = 1 - \varepsilon.$$

Let $\eta > \varepsilon_t$. For $x \in B(0, \eta)^c$ and $y \in B(0, \varepsilon_t)$, we can choose the minimal positive constant τ which satisfies

$$(3.4) \quad \frac{1 - \tau}{|x|^n} \leq \frac{1}{|x - y|^n} \leq \frac{1 + \tau}{|x|^n}.$$

Then $\tau \rightarrow 0_+$ as $t \rightarrow 0_+$.

Define $d\mu_t^1(x) = \chi_{B(0, \varepsilon_t)}(x)d\mu_t(x)$ and $d\mu_t^2(x) = \chi_{B(0, \varepsilon_t)^c}(x)d\mu_t(x)$, where χ_E is the characteristic function of E . Hence we have

$$|T_\Omega \mu_t^1(x)| - |T_\Omega \mu_t^2(x)| \leq |T_\Omega \mu_t(x)| \leq |T_\Omega \mu_t^1(x)| + |T_\Omega \mu_t^2(x)|.$$

For any given $\lambda > 0$, let

$$F_\lambda^t = \{x \in \mathbb{R}^n : |T_\Omega \mu_t(x)| > \lambda\},$$

$$F_{1, \lambda}^t = \{x \in \mathbb{R}^n : |T_\Omega \mu_t^1(x)| > \lambda\}$$

and

$$F_{2, \lambda}^t = \{x \in \mathbb{R}^n : |T_\Omega \mu_t^2(x)| > \lambda\}.$$

Since Ω satisfies the L^1 -Dini condition, by Lemma 3.4, T_Ω is of weak type (1,1). Therefore

$$(3.5) \quad \begin{aligned} m(F_{2, \delta\lambda}^t) &= m(\{x \in \mathbb{R}^n : |T_\Omega \mu_t^2(x)| > \delta\lambda\}) \leq \frac{C}{\delta\lambda} |\mu_t^2|(\mathbb{R}^n) \\ &= \frac{C}{\delta\lambda} |\mu_t|(B(0, \varepsilon_t)^c) \leq \frac{C\varepsilon}{\delta\lambda}. \end{aligned}$$

Since $F_{1, (1+\delta)\lambda}^t \subset F_{2, \delta\lambda}^t \cup F_\lambda^t$ and $F_\lambda^t \subset F_{2, \delta\lambda}^t \cup F_{1, (1-\delta)\lambda}^t$, by (3.5) we have the following estimate

$$(3.6) \quad -\frac{C\varepsilon}{\delta\lambda} + m(F_{1, (1+\delta)\lambda}^t) \leq m(F_\lambda^t) \leq \frac{C\varepsilon}{\delta\lambda} + m(F_{1, (1-\delta)\lambda}^t).$$

By the choice of ε and δ , $m(F_{1, (1+\delta)\lambda}^t)$ and $m(F_{1, (1-\delta)\lambda}^t)$ may approximate to $m(F_\lambda^t)$ as $t \rightarrow 0_+$ by (3.6). It is easy to see that

$$m(F_{1, (1+\delta)\lambda}^t) - \omega_n \eta^n \leq m(F_{1, (1+\delta)\lambda}^t \cap B(0, \eta)^c) \leq m(F_{1, (1+\delta)\lambda}^t)$$

where ω_n is the Lebesgue measure of unit ball in \mathbb{R}^n . Therefore we conclude that $m(F_{1, (1+\delta)\lambda}^t \cap B(0, \eta)^c)$ approximates to $m(F_{1, (1+\delta)\lambda}^t)$ as $\eta \rightarrow 0_+$. Similarly, $m(F_{1, (1-\delta)\lambda}^t \cap B(0, \eta)^c)$ approximates to $m(F_{1, (1-\delta)\lambda}^t)$ as $\eta \rightarrow 0_+$.

Now we split $T_\Omega \mu_t^1(x)$ into two parts:

$$T_\Omega \mu_t^1(x) = \lim_{\varepsilon' \rightarrow 0_+} \int_{|x-y| > \varepsilon'} \frac{\Omega(x)}{|x|^n} d\mu_t^1(y) + \lim_{\varepsilon' \rightarrow 0_+} \int_{|x-y| > \varepsilon'} \left(\frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n} \right) d\mu_t^1(y).$$

Using the triangle inequality, we have

$$\begin{aligned}
 (3.7) \quad & \left| \int_{|x-y|>\varepsilon'} \frac{\Omega(x)}{|x|^n} d\mu_t^1(y) \right| - \int_{|x-y|>\varepsilon'} \left| \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n} \right| d|\mu_t^1|(y) \\
 & \leq \left| \int_{|x-y|>\varepsilon'} \frac{\Omega(x-y)}{|x-y|^n} d\mu_t^1(y) \right| \\
 & \leq \left| \int_{|x-y|>\varepsilon'} \frac{\Omega(x)}{|x|^n} d\mu_t^1(y) \right| + \int_{|x-y|>\varepsilon'} \left| \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n} \right| d|\mu_t^1|(y).
 \end{aligned}$$

Denote

$$G_t := \left\{ x \in B(0, \eta)^c : \lim_{\varepsilon' \rightarrow 0+} \int_{|x-y|>\varepsilon'} \left| \frac{\Omega(x)}{|x|^n} - \frac{\Omega(x-y)}{|x-y|^n} \right| d|\mu_t^1|(y) \geq 2\delta\lambda \right\}.$$

Since

$$\left| \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n} \right| \leq \frac{|\Omega(x-y) - \Omega(x)|}{|x-y|^n} + |\Omega(x)| \left| \frac{1}{|x-y|^n} - \frac{1}{|x|^n} \right|,$$

we have

$$G_t \subset G_{t,1} \cap G_{t,2},$$

where

$$G_{t,1} := \left\{ x \in B(0, \eta)^c : \lim_{\varepsilon' \rightarrow 0+} \int_{|x-y|>\varepsilon'} \frac{|\Omega(x-y) - \Omega(x)|}{|x-y|^n} d|\mu_t^1|(y) \geq \delta\lambda \right\}$$

and

$$G_{t,2} := \left\{ x \in B(0, \eta)^c : \lim_{\varepsilon' \rightarrow 0+} \int_{|x-y|>\varepsilon'} |\Omega(x)| \left| \frac{1}{|x-y|^n} - \frac{1}{|x|^n} \right| d|\mu_t^1|(y) \geq \delta\lambda \right\}.$$

Consider $G_{t,1}$ firstly. If $x \in B(0, \eta)^c$ and $y \in B(0, \varepsilon_t)$, then $|x| > |y|$ and $\frac{1}{|x-y|^n} \leq \frac{1+\tau}{|x|^n}$ by (3.4). Using Chebychev's inequality, Fubini's theorem and making a polar transform, we have

$$\begin{aligned}
 m(G_{t,1}) & \leq m \left(\left\{ x \in B(0, \eta)^c : \int_{\mathbb{R}^n} \frac{|\Omega(x) - \Omega(x-y)|}{|x|^n} d|\mu_t^1|(y) \geq \frac{\delta\lambda}{1+\tau} \right\} \right) \\
 & \leq \frac{1+\tau}{\lambda\delta} \int_{B(0, \eta)^c} \int_{\mathbb{R}^n} \frac{|\Omega(x-y) - \Omega(x)|}{|x|^n} d|\mu_t^1|(y) dx \\
 & = \frac{1+\tau}{\lambda\delta} \int_{\mathbb{R}^n} \int_{B(0, \eta)^c} \frac{|\Omega(x-y) - \Omega(x)|}{|x|^n} dx d|\mu_t^1|(y) \\
 & = \frac{1+\tau}{\lambda\delta} \int_{\mathbb{R}^n} \int_{\eta}^{+\infty} \int_{\mathbb{S}^{n-1}} \left| \Omega\left(\theta - \frac{y}{r}\right) - \Omega(\theta) \right| d\sigma(\theta) \cdot \frac{dr}{r} d|\mu_t^1|(y)
 \end{aligned}$$

By Theorem 2.5, the L^1 -Dini condition in Definition 2.3 and Definition 1.1 are equivalent. So in the following we use the L^1 -Dini condition in Definition 2.3. Set $A(r) := \int_0^r \frac{\tilde{\omega}_1(s)}{s} ds$. Since Ω

satisfies the L^1 -Dini condition, we have $A(r) \rightarrow 0$ as $r \rightarrow 0_+$. Therefore

$$\begin{aligned}
 m(G_{t,1}) &\leq \frac{(1+\tau)}{\lambda\delta} \int_{\mathbb{R}^n} \int_{\eta}^{+\infty} \frac{\tilde{\omega}_1(|y|/r)}{r} dr d|\mu_t^1|(y) \\
 &= \frac{(1+\tau)}{\lambda\delta} \int_{\mathbb{R}^n} \int_0^{|y|/\eta} \frac{\tilde{\omega}_1(s)}{s} ds d|\mu_t^1|(y) \\
 &\leq \frac{(1+\tau)}{\delta\lambda} \int_0^{\varepsilon_t/\eta} \frac{\tilde{\omega}_1(s)}{s} ds \int_{\mathbb{R}^n} d|\mu_t^1|(y) \\
 &\leq \frac{(1+\tau)}{\delta\lambda} A(\varepsilon_t/\eta),
 \end{aligned} \tag{3.8}$$

where in the second equality we make a transform $|y|/r = s$.

Estimate of $m(G_{t,2})$ is similar to that of $m(G_{t,1})$. Again by using Chebychev's inequality, Fubini's theorem, (3.4) and making a polar transform, we have

$$\begin{aligned}
 m(G_{t,2}) &\leq \frac{1}{\delta\lambda} \int_{B(0,\eta)^c} \int_{\mathbb{R}^n} |\Omega(x)| \left| \frac{1}{|x|^n} - \frac{1}{|x-y|^n} \right| d|\mu_t^1|(y) dx \\
 &\leq \frac{1}{\delta\lambda} \int_{\mathbb{R}^n} \int_{B(0,\eta)^c} |\Omega(x)| \frac{(1+\tau)n|y|}{|x|^{n+1}} dx d|\mu_t^1|(y) \\
 &\leq \frac{(1+\tau)n}{\delta\lambda} \|\Omega\|_1 \int_{\mathbb{R}^n} \int_{\eta}^{\infty} \frac{dr}{r^2} |y| d|\mu_t^1|(y) \\
 &\leq \frac{(1+\tau)n\varepsilon_t}{\delta\lambda\eta} \|\Omega\|_1 |\mu_t^1|(\mathbb{R}^n) \\
 &\leq \frac{(1+\tau)n\varepsilon_t}{\delta\lambda\eta} \|\Omega\|_1,
 \end{aligned} \tag{3.9}$$

where in the fourth inequality we use $d\mu_t^1 = \chi_{B(0,\varepsilon_t)} d\mu_t$. Therefore combining these estimates for $G_{t,1}$ and $G_{t,2}$, we have

$$m(G_t) \leq m(G_{t,1}) + m(G_{t,2}) \leq \frac{(1+\tau)}{\delta\lambda} A(\varepsilon_t/\eta) + \frac{(1+\tau)n\varepsilon_t}{\delta\lambda\eta} \|\Omega\|_1. \tag{3.10}$$

It is easy to see that

$$\begin{aligned}
 m(\{x \in B(0,\eta)^c \cap G_t^c : |T_{\Omega}\mu_t^1(x)| > \lambda\}) &\leq m(\{F_{1,\lambda}^t \cap B(0,\eta)^c\}) \\
 &\leq m(\{x \in B(0,\eta)^c \cap G_t^c : |T_{\Omega}\mu_t^1(x)| > \lambda\}) + m(G_t).
 \end{aligned}$$

So if $x \in B(0,\eta)^c \cap G_t^c$, by the definition of G_t and (3.7),

$$\frac{|\Omega(x)|}{|x|^n} |\mu_t^1(\mathbb{R}^n)| - 2\delta\lambda \leq |T_{\Omega}\mu_t^1(x)| \leq \frac{|\Omega(x)|}{|x|^n} |\mu_t^1(\mathbb{R}^n)| + 2\delta\lambda.$$

Therefore we have

$$\begin{aligned}
 &\left\{ x \in B(0,\eta)^c \cap G_t^c : |T_{\Omega}\mu_t^1(x)| > (1-\delta)\lambda \right\} \\
 &\subset \left\{ x \in B(0,\eta)^c \cap G_t^c : \frac{|\Omega(x)|}{|x|^n} |\mu_t^1(\mathbb{R}^n)| > (1-3\delta)\lambda \right\}
 \end{aligned} \tag{3.11}$$

and

$$(3.12) \quad \begin{aligned} & \left\{ x \in B(0, \eta)^c \cap G_t^c : |T_\Omega \mu_t^1(x)| > (1 + \delta)\lambda \right\} \\ & \supset \left\{ x \in B(0, \eta)^c \cap G_t^c : \frac{|\Omega(x)|}{|x|^n} |\mu_t^1(\mathbb{R}^n)| > (1 + 3\delta)\lambda \right\}. \end{aligned}$$

By the definition of μ_t^1 ,

$$|\mu_t^1(\mathbb{R}^n)| = |\mu(\mathbb{R}^n) - \mu_t(B(0, \varepsilon_t)^c)|.$$

Note that $|\mu_t(B(0, \varepsilon_t)^c)| \leq |\mu_t|(B(0, \varepsilon_t)^c) \leq \varepsilon$, so we have

$$|\mu(\mathbb{R}^n)| - \varepsilon < |\mu_t^1(\mathbb{R}^n)| \leq |\mu(\mathbb{R}^n)| + \varepsilon.$$

Using (3.10), (3.11), (3.12) and Lemma 3.3 with $\alpha = 0$, we have

$$(3.13) \quad \begin{aligned} & m(F_{1, (1+\delta)\lambda}^t) \\ & \geq m(\{x \in B(0, \eta)^c \cap G_t^c : |T_\Omega \mu_t^1(x)| > (1 + \delta)\lambda\}) \\ & \geq m\left(\left\{x \in B(0, \eta)^c \cap G_t^c : \frac{|\Omega(x)|}{|x|^n} |\mu_t^1(\mathbb{R}^n)| \geq (1 + 3\delta)\lambda\right\}\right) \\ & \geq m\left(\left\{x \in \mathbb{R}^n : \frac{|\Omega(x)|}{|x|^n} |\mu_t^1(\mathbb{R}^n)| > (1 + 3\delta)\lambda\right\}\right) - \omega_n \eta^n - m(G_t) \\ & \geq \frac{\|\Omega\|_1}{n} \cdot \frac{|\mu(\mathbb{R}^n)| - \varepsilon}{(1 + 3\delta)\lambda} - \omega_n \eta^n - \frac{(1 + \tau)}{\delta\lambda} A\left(\frac{\varepsilon_t}{\eta}\right) - \frac{(1 + \tau)n\varepsilon_t}{\delta\lambda\eta} \|\Omega\|_1 \end{aligned}$$

and

$$(3.14) \quad \begin{aligned} & m(F_{1, (1-\delta)\lambda}^t) \\ & \leq m(\{x \in B(0, \tau)^c \cap G_t^c : |T_\Omega \mu_t^1(x)| > (1 - \delta)\lambda\}) + m(B(0, \eta)) + m(G_t) \\ & \leq m\left(\left\{x \in \mathbb{R}^n : \frac{|\Omega(x)|}{|x|^n} |\mu_t^1(\mathbb{R}^n)| > (1 - 3\delta)\lambda\right\}\right) + \omega_n \eta^n + m(G_t) \\ & \leq \frac{\|\Omega\|_1}{n} \cdot \frac{|\mu(\mathbb{R}^n)| + \varepsilon}{(1 - 3\delta)\lambda} + \omega_n \eta^n + \frac{(1 + \tau)}{\delta\lambda} A\left(\frac{\varepsilon_t}{\eta}\right) + \frac{(1 + \tau)n\varepsilon_t}{\delta\lambda\eta} \|\Omega\|_1. \end{aligned}$$

Here ω_n is the volume of unit ball in \mathbb{R}^n . Combining the above estimates (3.13), (3.14) and (3.5), we have

$$\begin{aligned} m(F_\lambda^t) & \geq m(F_{1, (1+\delta)\lambda}^t) - m(F_{2, \delta\lambda}^t) \\ & \geq \frac{\|\Omega\|_1}{n} \frac{|\mu(\mathbb{R}^n)| - \varepsilon}{(1 + 3\delta)\lambda} - \omega_n \eta^n - \frac{(1 + \tau)}{\delta\lambda} A\left(\frac{\varepsilon_t}{\eta}\right) - \frac{(1 + \tau)n\varepsilon_t}{\delta\lambda\eta} \|\Omega\|_1 - \frac{C\varepsilon}{\delta\lambda} \end{aligned}$$

and

$$\begin{aligned} m(F_\lambda^t) & \leq m(F_{1, (1-\delta)\lambda}^t) + m(F_{2, \delta\lambda}^t) \\ & \leq \frac{\|\Omega\|_1}{n} \frac{|\mu(\mathbb{R}^n)| + \varepsilon}{(1 - 3\delta)\lambda} + \omega_n \eta^n + \frac{(1 + \tau)}{\delta\lambda} A\left(\frac{\varepsilon_t}{\eta}\right) + \frac{(1 + \tau)n\varepsilon_t}{\delta\lambda\eta} \|\Omega\|_1 + \frac{C\varepsilon}{\delta\lambda}. \end{aligned}$$

Let $t \rightarrow 0_+$, then $\varepsilon_t \rightarrow 0_+$ and $\tau \rightarrow 0_+$. So $A(\frac{\varepsilon_t}{\eta}) \rightarrow 0_+$. Thus we obtain

$$\liminf_{t \rightarrow 0_+} m(F_\lambda^t) \geq \frac{\|\Omega\|_1}{n} \frac{|\mu(\mathbb{R}^n)| - \varepsilon}{(1 + 3\delta)\lambda} - \omega_n \eta^n - \frac{C\varepsilon}{\delta\lambda}$$

and

$$\limsup_{t \rightarrow 0_+} m(F_\lambda^t) \leq \frac{\|\Omega\|_1}{n} \frac{|\mu(\mathbb{R}^n)| + \varepsilon}{(1 - 3\delta)\lambda} + \omega_n \eta^n + \frac{C\varepsilon}{\delta\lambda}.$$

Note that $\varepsilon \leq \frac{1}{2}\delta\lambda$. Now let $\varepsilon \rightarrow 0_+$ firstly and $\delta \rightarrow 0_+$ secondly. Lastly let $\eta \rightarrow 0_+$. Then we have

$$\frac{\|\Omega\|_1 |\mu(\mathbb{R}^n)|}{n\lambda} \leq \liminf_{t \rightarrow 0_+} m(F_\lambda^t) \leq \limsup_{t \rightarrow 0_+} m(F_\lambda^t) \leq \frac{\|\Omega\|_1 |\mu(\mathbb{R}^n)|}{n\lambda}.$$

Thus we complete the proof. \square

3.3. The proof of Theorem 1.2. We write $T_\Omega \mu_t(x)$ as

$$\begin{aligned} (3.15) \quad T_\Omega \mu_t(x) &= \lim_{\epsilon \rightarrow 0_+} \int_{|x-y| > \epsilon} \frac{\Omega(x-y)}{|x-y|^n} d\mu_t(y) \\ &= \frac{1}{t^n} \lim_{\epsilon \rightarrow 0_+} \int_{|\frac{x}{t} - \frac{y}{t}| > \epsilon} \frac{\Omega(\frac{x}{t} - \frac{y}{t})}{|\frac{x}{t} - \frac{y}{t}|^n} d\mu(\frac{y}{t}) = \frac{1}{t^n} T_\Omega \mu(\frac{x}{t}). \end{aligned}$$

Then by (3.15), we have

$$\begin{aligned} m(\{x \in \mathbb{R}^n : |T_\Omega \mu_t(x)| > \lambda\}) &= m\left(\left\{x \in \mathbb{R}^n : \frac{1}{t^n} |T_\Omega \mu(\frac{x}{t})| > \lambda\right\}\right) \\ &= t^n m(\{x \in \mathbb{R}^n : |T_\Omega \mu(x)| > \lambda t^n\}). \end{aligned}$$

Applying Lemma 3.5, we get

$$\begin{aligned} \lim_{\lambda \rightarrow 0_+} \lambda m(\{x \in \mathbb{R}^n : |T_\Omega \mu(x)| > \lambda\}) &= \lim_{t \rightarrow 0_+} \lambda t^n m(\{x \in \mathbb{R}^n : |T_\Omega \mu(x)| > \lambda t^n\}) \\ &= \lim_{t \rightarrow 0_+} \lambda m(\{x \in \mathbb{R}^n : |T_\Omega \mu_t(x)| > \lambda\}) \\ &= \frac{1}{n} \|\Omega\|_1 |\mu(\mathbb{R}^n)|. \end{aligned}$$

Hence we complete the proof of Theorem 1.2. \square

4. PROOF OF THEOREM 1.5

In this section, we give the proof of Theorem 1.5. The proof is quite similar to that of Theorem 1.2. So we shall be brief and only indicate necessary modifications here. We first set up a result for $T_{\Omega, \alpha}$ which is similar to Lemma 3.5.

Lemma 4.1. *Set $0 < \alpha < n$ and $r = \frac{n}{n-\alpha}$. Let μ be an absolutely continuous signed measure with respect to Lebesgue measure on \mathbb{R}^n and $|\mu|(\mathbb{R}^n) < +\infty$. Suppose Ω satisfies (1.1), (1.2) and the L_α^r -Dini condition. Then we have*

$$(4.1) \quad \lim_{t \rightarrow 0_+} \lambda^r m(\{x \in \mathbb{R}^n : |T_{\Omega, \alpha} \mu_t(x)| > \lambda\}) = \frac{1}{n} \|\Omega\|_r^r |\mu(\mathbb{R}^n)|^r.$$

for any $\lambda > 0$.

Proof. The proof is similar to that of Lemma 3.5. Choose the same constants δ , ε , a_ε and ε_t as we do in the proof of Lemma 3.5. For the constant τ we choose the minimal constant such that

$$\frac{1-\tau}{|x|^{n-\alpha}} \leq \frac{1}{|x-y|^{n-\alpha}} \leq \frac{1+\tau}{|x|^{n-\alpha}}.$$

Since $T_{\Omega,\alpha}$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{\frac{n}{n-\alpha},\infty}$ (see Page.224 in [4]), we can get the similar estimate in (3.5). For the estimate similar to $m(G_{t,1})$, by Theorem 2.8, we use the equivalent L_α^r -Dini condition in Definition 2.7. In the estimate similar to (3.13) and (3.13'), we can use Lemma 3.3 with $0 < \alpha < n$. Proceeding the proof as we do in the proof of Lemma 3.5, we can finish the proof of Lemma 4.1. \square

Proof of Theorem 1.5. As we have done in the last part of section 3, we can establish the following dilation property of $T_{\Omega,\alpha}$ which is similar to (3.15):

$$T_{\Omega,\alpha}\mu_t(x) = \frac{1}{t^{n-\alpha}}T_{\Omega,\alpha}\mu\left(\frac{x}{t}\right).$$

By using above equality and Lemma 4.1, we have

$$\begin{aligned} & \lim_{\lambda \rightarrow 0_+} \lambda^r m(\{x \in \mathbb{R}^n : |T_{\Omega,\alpha}\mu(x)| > \lambda\}) \\ &= \lim_{t \rightarrow 0_+} (\lambda t^{n-\alpha})^r m(\{x \in \mathbb{R}^n : |T_{\Omega,\alpha}\mu(x)| > \lambda t^{n-\alpha}\}) \\ &= \lim_{t \rightarrow 0_+} \lambda^r m\left(\left\{x \in \mathbb{R}^n : |T_{\Omega,\alpha}\mu\left(\frac{x}{t}\right)| > \lambda t^{n-\alpha}\right\}\right) \\ &= \lim_{t \rightarrow 0_+} \lambda^r m(\{x \in \mathbb{R}^n : |T_{\Omega,\alpha}\mu_t(x)| > \lambda\}) = \frac{1}{n} \|\Omega\|_r^r |\mu(\mathbb{R}^n)|^r. \end{aligned}$$

Hence we complete the proof of Theorem 1.5. \square

Acknowledgment. The authors would like to express their deep gratitude to the referee for his/her very careful reading, important comments and valuable suggestions.

REFERENCES

1. A. P. Calderón, M. Weiss and A. Zygmund, *On the existence of singular integrals*, Singular integrals(Proc. Sympos. Pure Math.) **10**, 56-73, Amer. Math. Soc., Providence, R.I. 1967.
2. A. P. Calderón and A. Zygmund, *On singular integrals*. Amer. J. Math., **78** (1956), 289-309.
3. A. P. Calderón and A. Zygmund, *A note on singular integrals*. Studia Math., **65** (1979), 77-87.
4. S. Chanillo, D. Watson and R. L. Wheeden, *Some integral and maximal operator related to starlike sets*, Studia Math. **107** (1993), 223-255.
5. Y. Ding and S. Lu, *Homogeneous fractional integrals with Hardy spaces*, Tohoku Math. J. **52** (2000), 153-162.
6. G. Folland, *Real analysis. Modern techniques and their applications*, Second edition. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley Sons, Inc., New York, 1999.
7. L. Grafakos, *Classic Fourier Analysis*, Graduate Texts in Mathematics, Vol. **249**, Springer, New York, 2008.
8. T. Iwaniec and G. Martin, *Riesz transforms and related singular integrals*. J. Reine Angew. Math. **473** (1996), 25-57.
9. P. Janakiraman, *Weak-type estimates for singular integrals and the Riesz transform*, Indiana. Univ. Math. J. **53** (2004), 533-555.

10. P. Janakiraman, *Limiting weak-type behavior for singular integral and maximal operators*, Trans. Amer. Math. Soc. **358** (2006), 1937-1952.
11. S. Lu, Y. Ding and D. Yan, *Singular integrals and related topics*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.
12. A. Seeger, *Singular integral operators with rough convolution kernels*, J. Amer. Math. Soc., **9** (1996), 95-105.
13. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*. Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J. 1970.

Yong Ding

SCHOOL OF MATHEMATICAL SCIENCES

BEIJING NORMAL UNIVERSITY

LABORATORY OF MATHEMATICS AND COMPLEX SYSTEMS (BNU), MINISTRY OF EDUCATION

BEIJING, 100875, P. R. OF CHINA

E-mail address: dingy@bnu.edu.cn

Xudong Lai(CORRESPONDING AUTHOR)

SCHOOL OF MATHEMATICAL SCIENCES

BEIJING NORMAL UNIVERSITY

LABORATORY OF MATHEMATICS AND COMPLEX SYSTEMS (BNU), MINISTRY OF EDUCATION

BEIJING, 100875, P. R. OF CHINA

E-mail address: xudonglai@mail.bnu.edu.cn